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AN ALGORITHM FOR SOLVING INTERVAL LINEAR PROGRAMMING PROBLEMS

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AN ALGORITHM FOR SOLVING INTERVAL LINEAR PROGRAMMING PROBLEMS*

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ABSTRACT

This paper presents an algorithm for solving interval linear programming problems. The algorithm is a finite iterative method, which in each iteration solves a full row rank interval linear programming problem, with only one additional constraint. The solution and/or problem chosen appears to be computationally more efficient than that in the Ben-Israel and Robers algorithm.

Introduction

The general linear programming problem (LP) may be defined as:

$$\text{(LP):} \qquad \text{Max } c^T x$$

$$\text{subject to}$$

$$Ax = b$$

$$x \geq 0$$

where the matrix A and the vectors b and c are given.

Every (LP) problem is equivalent to an interval linear programming problem (IP) of the form:

$$\text{(IP):} \qquad \text{Max } c^T x$$

$$\text{subject to}$$

$$b^- \leq \bar{A}x \leq b^+$$

where

$$\bar{A} = \begin{pmatrix} A \\ I \end{pmatrix} \qquad b^- = \begin{pmatrix} -be \\ 0 \end{pmatrix} \qquad b^+ = \begin{pmatrix} be \\ Ue \end{pmatrix}$$

and $U > 0$ is "sufficiently large" (e.g. non-Archimedean transcendental for the unbounded case).

Because of the above reason and since the constraint system in many linear programming problems arise in (IP) form (wholly or partly), special algorithms for (IP) problems are important, and may be more efficient for those particular problems than general algorithms for (LP) problems.

0. Preliminaries and notation

$\emptyset$ - the empty set

$\{x\}$ - the set consisting of the single element x

$\{x: f(x)\}$ - the set of x satisfying $f(\cdot)$

iff - if and only if

R^n - the n dimensional real vector space

For any $x, y \in R^n$:

$x \geq y$ - denotes $x_i \geq y_i \quad 1 \leq i \leq n$

$x \perp y$ - denotes $\sum_{i=1}^{n} x_i y_i = 0$

For any subspace L of R^n:

$L^\perp = \{y: x \perp y, \text{ for all } x \in L\}$ the orthogonal complement of L

$x+L$ - the manifold $\{x + l, l \in L\}$

$R^{m\times n}$ - the space of m x n real matrices

$R_r^{m\times n}$ - $\{x \in R^{m\times n}; \quad \text{rank } x = r\}$

I_n - the n x n identity matrix

e_i - the i^{th} column of I_n

$e = \sum_{i=1}^{n} e_i$

For any $A \in R^{m\times n}$:

A^T - the transpose of A

$R(A) = \{y \in R^n : y = Ax \text{ for some } x \in R^n\}$ the range of A

$N(A) = \{x \in R^n : Ax = 0\}$ the null space of A

$A^\#$ = a right inverse of the matrix A

An interval linear programming (IP) is defined as

$$\text{(IP):} \qquad \text{Max } c^T x$$

$$\text{subject to}$$

$$b^- \leq Ax \leq b^+$$

where $c = (c_j)$, $b^- = (b_i^-)$, $b^+ = (b_i^+)$, $A = (a_{ij})$ $(1 \leq i \leq m;\ \ 1 \leq j \leq n)$ are given and $b^- \leq b^+$.

Any $x \in R^n$ satisfying (2) is called a feasible solution of (IP). If (IP) has feasible solutions and $\{\mathrm{Max}\ \ c^T x:\ \ x \text{ feasible}\}$ is finite then (IP) is bounded.

Lemma 1 [1]: A feasible (IP) is bounded iff $c \in N(A)^{\perp}$

Lemma 2 [1]: If A is of full row rank then the optimal solution x^* of (IP) is given by:

$$x^* = \sum_{i \in I^-} b_i^- A_i^{\#} + \sum_{i \in I^+} b_i^+ A_i^{\#} + \sum_{i \in I^0} [\theta_i b_i^+ + (1 - \theta_i) b_i^-] A_i^{\#} + N(A)$$

where

$$I^- = \{i:\ c^T A_i^{\#} < 0\}$$

$$I^+ = \{i:\ c^T A_i^{\#} > 0\}$$

$$I^0 = \{i:\ c^T A_i^{\#} = 0\}$$

$$0 \leq \theta_i \leq 1, \quad i \in I^0$$

and

$$A_i^{\#} = \text{ith column of } A^{\#}$$

In this paper an algorithm for solving (IP) problems in the general case where A is not of full row rank will be described. We shall first solve a special class of (IP) problems where the coefficient matrix can be separated into $A = \begin{pmatrix} F \\ h^T \end{pmatrix}$ where F is a full row rank matrix, and h^T is a single vector.

This result forms the basis for our algorithm to solve the general (IP) problem, with coefficient matrix $\begin{bmatrix} F \\ H \end{bmatrix}$, where H is any matrix. See also Ben-Israel and Robers [6], [7].

1. The Subproblem

In order to solve the general (IP) problem we will describe in section 2 an iterative algorithm which will in each step make use of the method described below to solve subproblems of the form:

(1) $$\text{Max } c^T x$$

subject to

(2) $$d^- \leq Fx \leq d^+$$

(3) $$g^- \leq h^T x \leq g^+$$

where F is of full row rank.

Let us substitute

(4) $$z = Fx$$

in (1), (2), (3) in order to obtain

(5) $$\text{Max } c^T F^{\#} z$$

subject to

(6) $$d^- \leq z \leq d^+$$

(7) $$g^- \leq h^T F^{\#} z \leq g^+$$

where $F^{\#}$ is the right inverse of the matrix F.

Let

(8) $$\tilde{z}_i = \begin{cases} z_i - d_i^- & \text{if } h^T F_i^{\#} \geq 0 \\ d_i^+ - z_i & \text{if } h^T F_i^{\#} < 0 \end{cases}$$

Thus we have

(9) $$0 \leq \tilde{z}_i \leq d_i^+ - d_i^-$$

If $\{+\}$ refers to indices i for which $h^T F_i^{\#} > 0$, $\{0\}$ to those for which $h^T F_i^{\#} = 0$ and $\{-\}$ to indices for which $h^T F_i^{\#} < 0$, then (5) can be written equivalently as:

$$\text{Max} \sum_{+,0} c^T F_i^{\#}(\tilde{z}_i + d_i^-) + \sum_{-} c^T F_i^{\#}(d_i^+ - \tilde{z}_i) =$$

(10) $$\text{Max} \sum_{+,0} c^T F_i^{\#} d_i^- + \sum_{-} c^T F_i^{\#} d_i^+ + \sum_{+,0} c^T F_i^{\#} \tilde{z}_i + \sum_{-} (-1) c^T F_i^{\#} \tilde{z}_i$$

Let us denote by

(11) $$\eta(\tilde{z}) = \sum_{+,0} h^T F_i^{\#}(\tilde{z}_i + d_i^-) + \sum_{-} h^T F_i^{\#}(d_i^+ - \tilde{z}_i) = h^T F^{\#} z$$

then

(12) $$\eta(0) = \sum_{+,0} h^T F_i^{\#} d_i^- + \sum_{-} h^T F_i^{\#} d_i^+$$

Using (10),(11),(12) we obtain an equivalent problem to (5),(6),(7) of the form:

$$\text{(13)} \qquad \operatorname{Max} \sum_{+,0} c^T F_i^{\#} \tilde{z}_i + \sum_{-} (-1) c^T F_i^{\#} \tilde{z}_i$$

subject to

$$\text{(14)} \qquad 0 \le \tilde{z}_i \le d_i^+ - d_i^-$$

$$\text{(15)} \qquad g^- - n(0) \le \sum_{i=1}^{n} |h^T F_i^{\#}| \tilde{z}_i \le g^+ - n(0) \qquad \text{where } n = \operatorname{rank}(F)$$

The system (14),(15) will be inconsistent iff either

$$\text{(16)} \qquad g^+ < n(0)$$

or

$$\text{(17)} \qquad n(d^+ - d^-) < g^-$$

and we have

<u>Lemma 1</u>: Necessary and sufficient conditions for problem (13),(14),(15) to be feasible is that both $n(0) \le g^+$ and $n(d^+ - d^-) \ge g^-$.

<u>Proof</u>: Follows directly by substituting the upper and lower bounds of $\tilde{z}$ in the additional constraint (15).

Let γ_i denote the ratio of the coefficients of $\tilde{z}_i$ in (13) and (15), then γ_i can be written as:

$$\text{(18)} \qquad \gamma_i = \begin{cases} \dfrac{c^T F_i^{\#}}{|h^T F_i^{\#}|} & i \in \{+\} \\[2ex] \dfrac{-c^T F_i^{\#}}{|h^T F_i^{\#}|} & i \in \{-\} \\[2ex] M & i \in \{0\} \text{ and } c^T F_i^{\#} \ge 0 \\[1ex] -M & i \in \{0\} \text{ and } c^T F_i^{\#} < 0 \end{cases}$$

where M is a positive number of dominating magnitude.

(At this point we may write down the optimal values of $\tilde{z}_i$ for $i \in \{0\}$:

$$i \in \{0\} \Rightarrow \tilde{z}_i^* = \begin{cases} d_i^+ - d_i^- & \text{if } c^T F_i^\# > 0 \\ 0 & \text{if } c^T F_i^\# \le 0) . \end{cases}$$

Let

(19)
$$\alpha_i = \begin{cases} c^T F_i^\# & i \in \{+,0\} \\ -c^T F_i^\# & i \in \{-\} \end{cases}$$

where $\{+,\ 0\} = \{+\} \cup \{0\}$,

and let $\{\gamma_{(i)}\}$ denote the set of γ_j's arranged in decreasing order, with $(j) < (k)$ also if $\gamma_j = \gamma_k$ but the original index of (j) is less than the original index of (k), where (i) is the i^{th} index order of $\{\gamma_{(i)}\}$.

Let

(20) P = Max integer $\ni \alpha_{(i)} \ge 0$;

$P = 0$ if $\alpha_{(1)} < 0$.

(21) P_1 = Least integer $\ni Sp_1 \equiv \sum_{i=1}^{P_1} |h^T F^\#_{(i)}| (d^+_{(i)} - d^-_{(i)}) > g^- - \eta(0)$

(22) P_2 = Least integer $\ni Sp_2 \equiv \sum_{i=1}^{P_2} |h^T F^\#_{(i)}| (d^+_{(i)} - d^-_{(i)}) > g^+ - \eta(0)$

If $S_n = \sum_{i=1}^{n} |h^T F^\#_{(i)}| (d^+_{(i)} - d^-_{(i)}) \not> g^+ - \eta(0)$, set $P_2 = n + 1$.

If $S_n = g^- - \eta(0)$, set $P_1 = n + 1$. If $S_n < g^- - \eta(0)$, the problem is inconsistent.

Remark 1

Note that since the denominator of γ_j is always positive, the elements in $\{\gamma_{(i)}\}$ are ordered in such a way that those with positive values of $\alpha_{(i)}$ appear first, and those with negative values later.

Remark 2

The assumption $c^T F_i^{\#} \neq 0$ can always be made if desired since, if not, a perturbation, essentially equivalent to that introduced by Charnes[2] for the simplex method, can be performed. The perturbed problem is obtained from the original problem by replacing the vector $c^T F^{\#}$ by the perturbed vector

$$(c_\varepsilon)_i = c^T F_i^{\#} + \varepsilon^i \tag{23}$$

where ε is a sufficiently small and positive number, so that optimal solutions to the perturbed problem are optimal solutions of the original problem and $(c_\varepsilon)_i \neq 0$ $(i = 1,\ldots,n)$.

The optimal solution $\tilde{z}^*$ to (13),(14),(15) can now be written for the different cases in the following manner:

Case I: For $P < P_1 \leq P_2$ the optimal solution is:

$$\tilde{z}^*_{(i)} = \begin{cases} d^+_{(i)} - d^-_{(i)} & 1 \leq i \leq P_1-1 \\ \theta_{(P_1)} = [g^- - \eta(0) - \sum_{i=1}^{P_1-1} |h^T F^{\#}_{(i)}|(d^+_{(i)} - d^-_{(i)})]/|h^T F^{\#}_{(P_1)}| & i = P_1 \\ 0 & P_1+1 \leq i \leq n \end{cases} \tag{24}$$

Case II: For $P_1 \leq P_2 \leq P$ the optimal solution is:

$$\tilde{z}^*_{(i)} = \begin{cases} d^+_{(i)} - d^-_{(i)} & 1 \leq i \leq P_2-1 \\ \theta_{(P_2)} = [g^+ - \eta(0) - \sum_{i=1}^{P_2-1} |h^T F^{\#}_{(i)}|(d^+_{(i)} - d^-_{(i)})]/|h^T F^{\#}_{(P_2)}| & i = P_2 \\ 0 & P_2+1 \leq i \leq n \end{cases} \tag{25}$$

Case III: For $P_1 \le P < P_2$ the optimal solution is:

$$\text{(26)} \qquad \tilde{z}^*_{(i)} = \begin{cases} d^+_{(i)} - d^-_{(i)} & 1 \le i \le P \\ 0 & P+1 \le i \le n \end{cases}$$

Remark: In cases I and II above, the denominator of the expression defining Θ will not be zero. The reason for this is as follows:

Case I: $h^T F^\#_{(P_1)} = 0 \Rightarrow P_1 = 1$, since zero is a lower bound for

$Sp_1 \equiv \sum_{i=1}^{P_1} |h^T F^\#_{(i)}| (d^+_{(i)} - d^-_{(i)})$ and P_1 is the least integer for which $Sp_1 > g^- - \eta(0)$.

But $h^T F^\#_{(1)} = 0$ also implies that $\alpha_{(1)} \ge 0$, from the definition of $\gamma_{(i)}$. i.e., $P \ge 1$.

This contradicts $P < P_1$ for case I.

Case II: $h^T F^\#_{(P_2)} = 0 \Rightarrow P_2 = 1$ (as above)

$$\Rightarrow Sp_2 = 0.$$

But $g^+ - \eta(0) \ge 0$ is a necessary condition for consistency of (14), (15). Hence $Sp_2 \not> g^+ - \eta(0)$. This contradicts the definition of P_2.

The corresponding optimal values of $z^*(i)$ for the three cases are:

Case I:

$$\text{(27)} \qquad z^*_{(i)} = \begin{cases} d^+_{(i)} & (i) \in \{+,0\} & 1 \le i \le P_1 - 1 \\ d^-_{(i)} & (i) \in \{-\} & 1 \le i \le P_1 - 1 \\ \theta_{(P_1)} + d^-_{(P_1)} & (i) \in \{+\} & i = P_1 \\ d^+_{(P_1)} - \Theta_{(P_1)} & (i) \in \{-\} & i = P_1 \\ d^-_{(i)} & (i) \in \{+,0\} & P_1 + 1 \le i \le n \\ d^+_{(i)} & (i) \in \{-\} & P_1 + 1 \le i \le n \end{cases}$$

Case II:

$$(28)\qquad z^*_{(i)}=\begin{cases} d^+_{(i)} & (i)\varepsilon\ \{+,0\} \qquad 1\le i\le P_2-1\\ d^-_{(i)} & (i)\varepsilon\ \{-\} \qquad 1\le i\le P_2-1\\ \theta_{(P_2)}+d^-_{(P_2)} & (i)\varepsilon\ \{+\} \qquad i=P_2\\ d^+_{(P_2)}-\theta_{(P_2)} & (i)\varepsilon\ \{-\} \qquad i=P_2\\ d^-_{(i)} & (i)\varepsilon\ \{+,0\} \qquad P_2+1\le i\le n\\ d^+_{(i)} & (i)\varepsilon\ \{-\} \qquad P_2+1\le i\le n \end{cases}$$

Case III:

$$z^*_{(i)}=\begin{cases} d^+_{(i)} & (i)\ \varepsilon\ \{+,0\} \qquad 1\le i\le P\\ d^-_{(i)} & (i)\ \varepsilon\ \{-\} \qquad 1\le i\le P\\ d^-_{(i)} & (i)\ \varepsilon\ \{+,0\} \qquad P+1\le i\le n\\ d^+_{(i)} & (i)\ \varepsilon\ \{-\} \qquad P+1\le i\le n \end{cases}$$

Remark 3

Observe that $\theta_{(P_1)}+d^-_{(P_1)}$ in (27) is independent of $d^-_{(P_1)}$ and $d^+_{(P_1)}$.

To show this we set

$$(30)\qquad \theta_{(P_1)}+d^-_{(P_1)}=[g^- -\sum_{+,0} h^TF^\#_i d^-_i - \sum_{-} h^TF^\#_i d^+_i - \sum_{i=1}^{P_1-1}|h^TF^\#_{(i)}|(d^+_{(i)}-d^-_{(i)}) + d^-_{(P_1)}h^TF^\#_{(P_1)}]/\ h^TF^\#_{(P_1)}$$

Since $P_1\ \varepsilon\ \{+,0\}$ we obtain from (30) that

$$(31)\qquad \theta_{(P_1)}+d^-_{(P_1)}=[g^- -\sum_{\substack{i\varepsilon\{+,0\}\\ i\ne(P_1)}} h^TF^\#_i d^-_i -\sum_{-} h^TF^\#_i d^+_i - \sum_{i=1}^{P_1-1}|h^TF^\#_{(i)}|(d^+_{(i)}-d^-_{(i)})]/h^TF^\#_{(P_1)}$$

which is independent of $d^-_{(P_1)}$ and $d^+_{(P_1)}$.

In the same way we get that $d^+_{(P_1)} - \theta_{(P_1)}$ is independent of $d^+_{(P_1)}$ and $d^-_{(P_1)}$ and that $\theta_{(P_2)} + d^-_{(P_2)}$ and $d^+_{(P_2)} - \theta_{(P_2)}$ are independent of $d^+_{(P_2)}$ and $d^-_{(P_2)}$.

For each of the three cases above the optimal solutions x^* for (5), (6), (7) are given by

(32) $$x^* = F^{\#} z^* + (I - F^{\#}F)y$$

where y is arbitrary.

<u>Example 1</u>

Solve $$\text{Max } x_1 + 2x_2$$

subject to

$$-9 \le -3x_1 + x_2 \le 9$$
$$0 \le x_2 \le 8$$
$$2 \le x_1 + x_2 \le 6$$

In this case we choose $F = \begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix}$. We have

$$F^{\#} = F^{-1} = \begin{pmatrix} -1/3 & 1/3 \\ 0 & 1 \end{pmatrix} \qquad c^T F^{-1} = (-1/3,\ 7/3)$$

$$h^T = (1,1) \qquad h^T F^{-1} = (-1/3,\ 4/3)$$

Thus the problem is transformed into

$$\text{Max } -1/3 z_1 + 7/3 z_2$$

subject to

$$-9 \le z_1 \le 9$$
$$0 \le z_2 \le 8$$
$$2 \le -1/3 z_1 + 4/3 z_2 \le 6$$

Now let
$$\tilde{z}_1 = 9 - z_1$$
$$\tilde{z}_2 = z_2$$

The problem to be solved is

$$\text{Max } -1/3(9 - \tilde{z}_1) + 7/3\tilde{z}_2 \sim \text{Max } 1/3\tilde{z}_1 + 7/3\tilde{z}_2$$

subject to

$$0 \le \tilde{z}_1 \le 18$$
$$0 \le \tilde{z}_2 \le 8$$
$$5 \le 1/3\tilde{z}_1 + 4/3\tilde{z}_2 \le 9$$

We have

$$\gamma_1 = \frac{1/3}{1/3} = 1 \qquad \gamma_2 = \frac{7/3}{4/3} = 7/4 \qquad \{\gamma_{(i)}\} = \{7/4, 1\}$$

$$\sum_{i=1}^{P_1} |h^T F^{\#}_{(i)}| (d^+_{(i)} - d^-_{(i)}) = 4/3 \cdot 8 = 10\tfrac{2}{3} > 9$$

Thus $P_1 = 1$, $P_2 = 1$, $P = 2$, and

$$\tilde{z}^*_i = \begin{pmatrix} 0 \\ \frac{9}{4/3} \end{pmatrix} = \begin{pmatrix} 0 \\ 27/4 \end{pmatrix}$$

$$z^*_i = \begin{pmatrix} 9 \\ 27/4 \end{pmatrix} \qquad x^* = F^{-1}z^* = \begin{pmatrix} -1/3 & 1/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ 27/4 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 27/4 \end{pmatrix}$$

<u>Theorem 1</u>

If feasible, an optimal solution to problem (5),(6),(7) is given for the three cases I, II, III by (27),(28),(29) respectively.

Proof: Follows directly from the way z_i^* was constructed.

Remark 4:

If (5),(6),(7) is feasible it is always possible to delete a constraint from the set of constraints (6),(7) without affecting the optimal solution. This follows from Remark 3 in which we showed that in Case I, z^* is independent of (P_1), thus the (P_1) constraint of F can be deleted. In Case II constraint (P_2) of F can be deleted, and in Case III, the additional constraint (7) can be deleted. In each of the above cases deletion of the suitable constraint will not change the optimal solution.

2 An Algorithm for Solving General (IP) Problems

In this section an algorithm for solving (IP) problems will be described. The (IP) problem to be solved is:

(33) $$\text{Max } c^T x$$
subject to

(34) $$b^- \leq Ax \leq b^+$$

where $b^- \leq b^+$, $c \perp N(A)$ and $A \in R_r^{m \times r}$. If $m = r$ then an optimal solution to (33),(34) is given by (29).

For $x \in R^{1 \times r}$ let

(35) $$I(x) = \{i: (Ax)_i > b_i^+ \text{ or } (Ax)_i < b_i^-\} .$$

Let $H(0)$ be any set of r indices from $\{1,\ldots,m\}$ such that $\{a_i: i \in H(0)\}$ is a linearly independent set of rows of A, and let

(36) $$i(0) \in \{1,\ldots,m\} \qquad i(0) \notin H(0)$$

Proceed to the first iteration:

Iteration $\nu \geq 1$: Apply the method in 1 to determine infeasibility or to find an optimal solution $x^*(\nu)$ to the following problem:

<u>Problem ν:</u>

(37) $$\text{Max } c^T x$$

subject to

(38) $$b_i^- \leq a_i x \leq b_i^+ \qquad i \in H(\nu-1)$$

(39) $$b_{i(\nu-1)}^- \leq a_{i(\nu-1)} x \leq b_{i(\nu-1)}^+$$

If problem ν is infeasible then (IP) is infeasible; stop.

Alternately, problem (33) (34) can be "regularized" a la Charnes-Cooper so that infeasibility is determined at optimal solution if that is the case.

If $I(x^*(\nu)) = \emptyset$ then $x^*(\nu)$ is an optimal solution of (IP); stop.

Otherwise let

(40) $$i(\nu) \in I(x^*(\nu))$$

For Case I we choose:

(41) $$H(\nu) = H(\nu-1) \cup \{i(\nu-1)\}/\{P_1(\nu)\}$$

where / denotes deletion.

For Case II

(42) $$H(\nu) = H(\nu-1) \cup \{i(\nu-1)\}/\{P_2(\nu)\}$$

(For the definition of $P_1(\nu)$ and $P_2(\nu)$ see (21),(22)). For Case III,

(42a) $$H(\nu) = H(\nu - 1).$$

proceed to iteration $\nu + 1$.

Notice that from the definition of P_1 and P_2 the matrix with rows a_i, $i \in H(\nu)$ is of full row rank. Moreover, since the coefficient matrix in iteration ν differs from the coefficient matrix in iteration ν-1 by only one row, the product form of the inverse (e.g. Charnes and Cooper [3]) may be used to compute the new inverse.

<u>Theorem 2</u>

The algorithm described above terminates in a finite number of steps either with the conclusion that (IP) is infeasible, or with an optimal solution to (IP).

Proof: Since in iteration ν the algorithm solves a full row rank problem with one additional constraint which was not satisfied by the optimal solution $x^*(\nu-1)$, it follows from Remark 2 that

$$c^T x^*(\nu-1) > c^T x^*(\nu) \tag{43}$$

which assures the finiteness.

The optimality is assured since at each step of the algorithm we find an optimal solution to a less restricted problem.

Example 2 Solve

$$\text{Max } x_1 + 2x_2$$
$$\text{subject to}$$
$$0 \le x_1 \le 6$$
$$0 \le x_2 \le 8$$
$$2 \le x_1 + x_2 \le 6$$
$$-9 \le -3x_1 + x_2 \le 9$$

Iteration 0:

$$H(0) = \{1,2\} \qquad i(0) = 3$$

Iteration 1: Solve

$$\text{Max } x_1 + 2x_2$$
$$\text{subject to}$$
$$0 \le x_1 \le 6$$
$$0 \le x_2 \le 8$$
$$2 \le x_1 + x_2 \le 6$$

$\{\gamma_{(i)}\} = \{2,1\}$ $\qquad P_1 = 1,\ P_2 = 1,\ P = 2$

From (24) $x^*(1) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$ which is the optimal solution since $I(x^*(1)) = \emptyset$.

Example 3 [5]

Solve

$$\text{Max } x_1 + 2x_2$$

subject to

$$0 \le x_1 \le 6$$
$$0 \le x_2 \le 8$$
$$-9 \le -3x_1 + x_2 \le 9$$
$$2 \le x_1 + x_2 \le 6 .$$

Iteration 0:

$$H(0) = \{1,2\} \qquad i(0) = 3$$

$$\tilde{z} = \begin{pmatrix} 6 - x_1 \\ x_2 \end{pmatrix}$$

Iteration 1:

Solve

$$\text{Max } -\tilde{z}_1 + 2\tilde{z}_2$$

subject to

$$0 \le \tilde{z}_1 \le 6$$
$$0 \le \tilde{z}_2 \le 8$$
$$+9 \le 3\tilde{z}_1 + \tilde{z}_2 \le 27$$

$$\{\gamma_{(i)}\} = \{2,-1/3\} \qquad P = 1 \qquad P_1 = 2 \qquad P_2 = 2$$

Thus $\tilde{z}^* = \begin{pmatrix} 1/3 \\ 8 \end{pmatrix}$, $x^*(1) = \begin{pmatrix} 17/3 \\ 8 \end{pmatrix}$, and $I(x^*(1)) = \{4\}$

Iteration 2:

Solve

$$\text{Max } x_1 + 2x_2$$

subject to

$$-9 \le -3x_1 + x_2 \le 9$$
$$0 \le x_2 \le 8$$
$$2 \le x_1 + x_2 \le 6$$

$$F(2) = \begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix} \quad F(2)^{-1} = \begin{pmatrix} -1/3 & 1/3 \\ 0 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} -1/3z_1 + 1/3z_2 \\ z_2 \end{pmatrix}$$

Thus substituting $F(2)x = z$ we obtain

$$\text{Max} \quad -1/3z_1 + 7/3z_2$$

subject to

$$-9 \le z_1 \le 9$$
$$0 \le z_2 \le 8$$
$$2 \le -1/3z_1 + 4/3z_2 \le 6$$

substituting $\tilde{z} = \begin{pmatrix} 9 - z_1 \\ z \end{pmatrix}$ we obtain:

$$\text{Max} \quad +1/3\tilde{z}_1 + 7/3\tilde{z}_2$$

subject to

$$0 \le \tilde{z}_1 \le 18$$
$$0 \le \tilde{z}_2 \le 8$$
$$5 \le 1/3\tilde{z}_1 + 4/3\tilde{z}_2 \le 9$$

$\{\gamma_{(i)}\} = \{\frac{7/3}{4/3}, 1\}$ $\quad P = 2 \quad P_1 = 1 \quad P_2 = 1$

$$\tilde{z}^* = \begin{pmatrix} 0 \\ \frac{9 \cdot 3}{4} \end{pmatrix} \quad z^* = \begin{pmatrix} 9 \\ \frac{27}{4} \end{pmatrix} \quad x^*(2) = \begin{pmatrix} -3 + \frac{27}{12} \\ \frac{27}{4} \end{pmatrix} = \begin{pmatrix} -3/4 \\ \frac{27}{4} \end{pmatrix}$$

and $I(x_2^*) = \{1\}$.

<u>Iteration 3</u>: Solve

$$\text{Max } x_1 + 2x_2$$

subject to

$$-9 \le -3x_1 + x_2 \le 9$$
$$2 \le x_1 + x_2 \le 6$$
$$0 \le x_1 \le 6$$

$$F(3) = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \qquad F(3)^{-1} = \begin{pmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \qquad x = \begin{pmatrix} -1/4z_1 + 1/4z_2 \\ 1/4z_1 + 3/4z_2 \end{pmatrix}$$

Thus substituting $F(3)x = z$ we obtain

$$\text{Max } -1/4z_1 + 1/4z_2 + 2/4z_1 + 6/4z_2 = \text{Max } 1/4z_1 + 7/4z_2$$

subject to

$$-9 \le z_1 \le 9$$
$$0 \le z_2 \le 6$$
$$0 \le -1/4z_1 + 1/4z_2 \le 6$$

substituting $\tilde{z} = \begin{pmatrix} 9-z_1 \\ z_2 \end{pmatrix}$ we obtain

$$\text{Max } 1/4(9-\tilde{z}_1) + 7/4\tilde{z}_2$$

$$0 \le \tilde{z}_1 \le 18$$
$$0 \le \tilde{z}_2 \le 6$$
$$9/4 \le 1/4\tilde{z}_1 + 1/4\tilde{z}_2 \le 6 + 9/4$$

$\{\gamma_{(i)}\} = \{7,-1\}$ $\quad P = 1 \quad P_1 = 2 \quad P_2 = 2$

$$\tilde{z}^* = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad z^* = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \quad x^*(3) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$I(x^*(3)) = \emptyset$. Thus $x^*(3)$ is optimal for (IP).

Observe that the number of iterations needed in order to solve Example 2 was strongly influenced by the choice of index from $I(\nu)$.

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